

THE LOWER LYAPUNOV EXPONENT OF HOLOMORPHIC MAPS

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ABSTRACT. For any polynomial map with a single critical point, we prove that its lower Lyapunov exponent at the critical value is negative if and only if the map has an attracting cycle. Similar statement holds for the exponential maps and some other complex dynamical systems.

1. INTRODUCTION AND MAIN RESULTS

In recent years, dynamical systems with different non-uniform hyperbolicity conditions have been studied. Speaking about one-dimensional (real or complex) dynamics, such restrictions are often put on the derivatives at critical values of the map (see last Section). Simplest and most studied are unicritical polynomial maps $f(z) = z^d + c$ and exponential maps $E(z) = a \exp(z)$. In this paper, we prove in particular that for each such polynomial or exponential map without sinks, but otherwise arbitrary, there is always a certain expansion along the critical orbit.

Theorem 1.1. *Let $f(z) = z^d + c$, where $d \geq 2$ and $c \in \mathbb{C}$. Assume that c does not belong to the basin of an attracting cycle. Then*

$$\chi_-(f, c) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(c)| \geq 0.$$

Theorem 1.1 has been known before for real c (more generally, for S-unimodal maps of an interval) [3].

Theorem 1.2. *Let $E(z) = a \exp(z)$, where $a \in \mathbb{C} \setminus \{0\}$. Assume that 0 does not belong to the basin of an attracting cycle. Then*

$$\chi_-(E, 0) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log |DE^n(0)| \geq 0.$$

These two theorems are special cases of the following theorem. Let $\mathcal{U} = \mathcal{U}_{V,V'}$ be the set of all holomorphic maps $f : V \rightarrow V'$ between open sets $V \subset V' \subset \mathbb{C}$, for which there exists a unique point $c = c(f) \in V'$ and a positive number $\rho = \rho(f)$ with the following properties:

Date: April 24, 2013.

The second author is partially supported by Polish MNiSW Grant N N201 607640. The third author is partially supported by Grant C-146-000-032-001 from National University of Singapore.

- (U1) $f : V \setminus f^{-1}(c) \rightarrow V' \setminus \{c\}$ is an unbranched covering map;
(U2) for each $n = 0, 1, \dots$, $f^n(c)$ is well defined and $B(f^n(c), \rho) \subset V'$.

Theorem 1.3. *For any $f \in \mathcal{U}_{V,V'}$, if $c(f)$ does not belong to the basin of an attracting cycle, then $\chi_-(c(f)) \geq 0$.*

See also last Section.

2. PROOF OF THEOREM 1.3

Let $f : V \rightarrow V'$ be a map in \mathcal{U} and let $c = c(f), \rho = \rho(f)$. Furthermore, let $AB(f)$ denote the union of the basin of attracting cycles of f . So when f has no attracting cycle, then $AB(f) = \emptyset$.

We need two lemmas for the proof of Theorem 1.3. In the first Lemma a general construction is introduced which is used also later on. Throughout the proofs, the Koebe principle applies.

Lemma 2.1. *Assume that c is not a periodic point. Given $\lambda > 1$ there exists δ_0 , such that for each $\delta \in (0, \delta_0)$, if $n \geq 1$ is the first return time of $z \in \overline{B(c; \delta)} \setminus AB(f)$ into $\overline{B(c; \delta)}$, then*

$$(2.1) \quad |Df^n(z)| \geq \lambda^{-n} \frac{|f^n(z) - c|}{\delta}.$$

Proof. Let $\delta \in (0, \rho/2]$. Let $n \geq 1$ and $z \in \overline{B(c; \delta)} \setminus AB(f)$ be as in the Lemma, and write $z_i = f^i(z)$. Let $\{\tau_i\}_{i=0}^n$ be a sequence of positive numbers with the following properties:

- (1) $\tau_n = |z_n - c|$ and $U_n = B(z_n, \tau_n)$;
- (2) for each $0 \leq i < n$, τ_i be the maximal number such that
 - $0 < \tau_i \leq \tau_{i+1}$ and
 - f^{n-i} maps a neighborhood U_i of z_i conformally onto $B(z_n, \tau_i)$.

Let

$$\mathcal{I} = \{0 \leq i < n : \tau_i < \tau_{i+1}\}$$

and let

$$N = \#\mathcal{I}.$$

Note that for each $i \in \mathcal{I}$, $c \in \partial f(U_i)$. Since f^n maps U_0 conformally onto $B(z_n, \tau_0)$, by the Koebe $\frac{1}{4}$ Theorem, we have

$$(2.2) \quad |Df^n(z)| \geq \frac{\tau_0}{4\varepsilon_0}, \text{ where } \varepsilon_0 = d(z_0, \partial U_0).$$

Claim 1. There exists a universal constant $K > 1$ such that for each $i \in \mathcal{I}$, we have $\tau_{i+1} \leq K\tau_i$. Moreover,

$$(2.3) \quad \varepsilon_0 \leq 3\delta.$$

Proof of Claim 1. We first note that for each $0 \leq i < n$, $U_i \not\supseteq \overline{B(c, 2\delta)}$ for otherwise,

$$f^{n-i}(U_i) = B(z_n, \tau_i) \subset \overline{B(c, 2\delta)} \subset U_i,$$

which implies by the Schwarz lemma that z_i , hence z_0 , is contained in the basin of an attracting cycle of f , a contradiction! Since $|z_0 - c| \leq \delta$, the inequality (2.3) follows.

Now let $i \in \mathcal{I}$. Then $i < n - 1$ and $U_{i+1} \supset \partial f(U_i) \ni c$, so

$$(2.4) \quad \text{diam}(f(U_i)) \geq |c - z_{i+1}| \geq \delta.$$

Since $U_{i+1} \not\ni \overline{B(c, 2\delta)}$, it follows that $\text{mod}(U_{i+1} \setminus f(U_i))$ is bounded from above by a universal constant. Since $f^{n-i-1} : U_{i+1} \rightarrow B(z_n, \tau_{i+1})$ is a conformal map, we have

$$\text{mod}(U_{i+1} \setminus f(U_i)) = \log \frac{\tau_{i+1}}{\tau_i}.$$

Thus τ_{i+1}/τ_i is bounded from above by a universal constant. \square

By (2.2), it follows that

$$(2.5) \quad |Df^n(z_0)| \geq \frac{|z_n - c|}{\delta} (12K^N)^{-1}.$$

Since c is not a periodic point, we have

$C(\delta) = \inf\{m \geq 1 : \exists z \in B(c, 2\delta) \text{ such that } f^m(z) \in B(c, 2\delta)\} \rightarrow \infty$ as $\delta \rightarrow 0$. Thus given $\lambda > 1$, there is $\delta_0 > 0$ such that when $\delta \in (0, \delta_0]$ we have

$$12K^N \leq \lambda^{C(\delta)(N+1)/2},$$

hence

$$(2.6) \quad |Df^n(z)| \geq \frac{|z_n - c|}{\delta} \lambda^{-C(\delta)(N+1)/2}.$$

To complete the proof, let us show that

$$(2.7) \quad n \geq \max(1, N)C(\delta) \geq C(\delta)(N+1)/2.$$

Indeed, for $i < i'$ in $\mathcal{I} \cup \{n-1\}$, we have $w := f^{n-i'-1}(c) \in B(c; 2\delta)$ and $f^{i'-i}(w) = f^{n-i-1}(c) \in B(c, 2\delta)$, $i' - i \geq C(\delta)$. Thus $n \geq C(\delta)N$. Moreover, since $z \in B(c; 2\delta)$ and $f^n(z) \in B(c, 2\delta)$ we have $n \geq C(\delta)$. The inequality follows.

Combining (2.6) and (2.7), we obtain the inequality (2.1). \square

Lemma 2.2. *Given $\lambda > 1$, $M_0 > 0$ and $\delta > 0$ there exists $\kappa = \kappa(M_0, \delta, \lambda)$ such that whenever $z \in \overline{B(c, M_0)} \setminus AB(f)$, $|f^j(z) - c| \geq \delta$ holds for $0 < j \leq n$ and $B(f^n(z), \delta) \subset V'$, we have*

$$|Df^n(z)| \geq \kappa \lambda^{-n}.$$

Proof. Fix λ and δ . We define the numbers τ_i , domains U_i , $0 \leq i \leq n$ and the number ε_0 as in the proof of Lemma 2.1, with the only difference that we start with $\tau_n = \delta$. Let us show that there exists $M = M(f)$ such that

$$(2.8) \quad U_i \not\ni \overline{B(c, M + \delta)} \text{ for each } 0 \leq i < n.$$

If $V \neq \mathbb{C}$, we define $M = d(c, \partial V)$. Then (2.8) is obvious since U_i must be in V .

If $V = \mathbb{C}$, then also $V' = \mathbb{C}$. In this case, $f : \mathbb{C} \rightarrow \mathbb{C}$ is either a polynomial or a transcendental entire function. It always has a (finite) periodic orbit P (this fact is trivial for polynomials, and was proved by Fatou for entire functions). Define $M = \max\{|w - c| : w \in P\}$. Then (2.8) holds, for otherwise, we would have that $B(z_n, \delta) = \overline{f^{n-i}(U_i)} \supset \overline{f^{n-i}(P)} = P$, hence, $B(z_n, \delta) \subset B(c, M + \delta)$. Then $\overline{f^{n-i}(U_i)} = \overline{B(z_n, \delta)} \subset \overline{B(c, M + \delta)} \subset U_i$, which would then imply $z_i \in AB(f)$ and hence $z \in AB(f)$, a contradiction.

Since $|z - c| \leq M_0$, it follows that

$$(2.9) \quad \varepsilon_0 \leq M + M_0 + \delta.$$

To complete the proof, we need to consider the following set of indexes

$$\mathcal{I}_\lambda = \{i \in \mathcal{I} : \lambda \tau_i \leq \tau_{i+1}\}.$$

For each $i \in \mathcal{I}_\lambda$, $\text{diam}(f(U_i)) \geq |z_{i+1} - c| \geq \delta$, so by the Koebe principle, there exists $\alpha = \alpha(\lambda) > 0$ such that

$$(2.10) \quad B(z_{i+1}, \alpha\delta) \subset f(U_i).$$

Moreover, by (2.8), there exists $K = K(\delta, \lambda) > 0$ such that

$$(2.11) \quad \frac{\tau_{i+1}}{\tau_i} = e^{\text{mod}(U_{i+1} \setminus f(U_i))} \leq K.$$

Furthermore, by (2.8) and $\tau_{i+1} \geq \lambda \tau_i$, there exists a constant $D = D(\delta, \lambda) > 0$ such that

$$(2.12) \quad |z_{i+1} - c| \leq D.$$

Let us prove that there exists $m_0 = m_0(\lambda, \delta)$ such that $\#\mathcal{I}_\lambda \leq m_0$. To this end, let $i(0) < i(1) < \dots < i(m-1)$ be all the elements of \mathcal{I}_λ . For each $0 \leq j \leq j' < m$, we have

$$\text{mod}(U_{i(j')+1} \setminus f^{i(j')-i(j)+1}(U_{i(j)})) = \log \frac{\tau_{i(j')+1}}{\tau_{i(j)}} \geq \lambda \cdot (j' - j + 1).$$

By (2.8) and (2.12), it follows that there exists $m_1 = m_1(\delta, \lambda)$ such that for $0 \leq j < j' < m$ with $j' - j \geq m_1$, $\text{diam}(f^{i(j')-i(j)+1}(U_{i(j)})) \leq \alpha\delta/2$. For such j, j' , since $z_{i(j)+1} \notin AB(f)$, $f^{i(j')-i(j)+1}(U_{i(j)})$ is not properly contained in $f(U_{i(j)})$, and thus by (2.10), we have $|z_{i(j)+1} - z_{i(j')+1}| \geq \alpha\delta/2$. In particular, the distance between any two distinct points in the set $\{z_{i(km_1)} : 0 \leq k < m/m_1\}$ is at least $\alpha\delta/2$. By (2.12), the last set is contained in a bounded set $\overline{B(c, D)}$, thus its cardinality is bounded from above by a constant. Thus $m = \#\mathcal{I}_\lambda$ is bounded from above.

It follows that

$$\tau_0 \geq \tau_n K^{-m_0} \lambda^{-(n-m_0)} \geq \delta K^{-m_0} \lambda^{-n}.$$

So

$$|Df^n(z)| \geq \frac{\tau_0}{4\epsilon_0} \geq \kappa \lambda^{-n},$$

where $\kappa = \delta K^{-m_0} (4(M + M_0 + \delta))^{-1}$. \square

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We may certainly assume that c is not periodic. Fix $\lambda > 1$, let δ_0 be given by Lemma 2.1 and let $\kappa = \kappa(\delta_0, \delta_0, \lambda)$ be given by Lemma 2.2. We shall show that for any $s \geq 1$ with $|f^s(c) - c| \leq \delta_0$, we have

$$(2.13) \quad |Df^s(c)| \geq \lambda^{-s}.$$

Once this is proved, by Lemma 2.2, it follows that for all $s \geq 1$, $|Df^s(c)| \geq \kappa \lambda^{-s}$, and thus $\chi_-(c) \geq -\lambda$, completing the proof of the theorem.

Now fix $s \geq 1$ with $|f^s(c) - c| \leq \delta_0$ and let us prove (2.13). To this end, define $\varepsilon_1 = \inf_{i=1}^s |f^i(c) - c|$ and let s_1 be minimal such that $|f^{s_1}(c) - c| = \varepsilon_1$. If $s_1 = s$ then we stop. Otherwise, define $\varepsilon_2 = \inf_{i=s_1+1}^s |f^i(c) - c|$ and let $s_2 \in \{s_1 + 1, \dots, s\}$ be minimal such that $|f^{s_2}(c) - c| = \varepsilon_2$. Repeating that argument, we obtain a sequence of positive numbers $\varepsilon_1 \leq \varepsilon_2 \leq \dots \leq \varepsilon_k \leq \delta_0$ and a sequence of integers $0 < s_1 < s_2 < \dots < s_k = s$.

Applying Lemma 2.1 to $z = c$, $\delta = \varepsilon_1$ and $n = s_1$, we obtain

$$|Df^{s_1}(c)| \geq \lambda^{-s_1}.$$

For each $i = 2, 3, \dots, k$, applying Lemma 2.1 to $z = f^{s_{i-1}}(c)$, $\delta = \varepsilon_i$ and $n = s_i - s_{i-1}$, we obtain

$$|Df^{s_i-s_{i-1}}(f^{s_{i-1}}(c))| \geq \lambda^{-(s_i-s_{i-1})}.$$

Therefore

$$|Df^s(c)| = |Df^{s_1}(c)| \prod_{i=2}^k |Df^{s_i-s_{i-1}}(f^{s_{i-1}}(c))| \geq \lambda^{-s}.$$

\square

3. SOME APPLICATIONS AND REMARKS

Let us note the following special case of Theorem 1.3:

Theorem 3.1. *Let g be a rational function on the Riemann sphere of degree at least two. Given a critical value c of g , define its postcritical set $P(c) = \overline{\cup_{n \geq 0} g^n(c)}$. Assume that c_0 is a critical value of g not in the basin of an attracting cycle, such that $P(c_0)$ is disjoint from the union X of the postcritical sets of all other critical values of g . Then*

$$\chi_-(g, c_0) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \|Dg^n(c_0)\| \geq 0,$$

where $\|\cdot\|$ denote the norm in the spherical metric.

Proof. By means of a Möbius conjugacy, we may assume that $\infty \in X$, so that the orbit of c_0 lies in a compact subset of \mathbb{C} and $\chi_-(g, c_0)$ can be calculated using the Euclidean metric instead of the spherical metric. Then define $V' = \overline{\mathbb{C}} \setminus X$ and $V = g^{-1}(V')$, and apply Theorem 1.3 to $g \in \mathcal{U}_{V,V'}$. \square

An immediate corollary of Theorem 1.1 along with Remark 13 of [1] is as follows:

Corollary 3.1. *Assume that the map $f(z) = z^d + c$ has no attracting cycles. Then the power series*

$$F(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{Df^n(c)}$$

has the radius of convergence at least 1, and

$$(3.1) \quad F(t) \neq 0 \text{ for every } |t| < 1.$$

Remark 3.2. The function $F(t)$ should be considered as the “Fredholm determinant” of the operator $T : \phi \mapsto \sum_{w:f(w)=z} \frac{\phi(w)}{Df(w)^2}$ acting in a space of functions ϕ , which are analytic outside of $J(f)$ and locally integrable on the plane. Then (3.1) reflects the fact that T is a contraction operator in this space. Note that this operator plays, in particular, an important role (after Thurston) in the problem of stability in holomorphic dynamics.

Another consequence of Theorem 1.3 is that:

Corollary 3.3. *Let $g_i : V_i \rightarrow V'_i$, $i = 0, 1$, be two mappings in the class \mathcal{U} which are quasi-conformally conjugated (i.e., there exists a q-c map $h : \mathbb{C} \rightarrow \mathbb{C}$ such that $h(V_0) = V_1$, $h(V'_0) = V'_1$, and $h \circ g_0 = g_1 \circ h$ on V_0). Assume that $\omega_{g_0}(c(g_0))$ (the ω -limit set of the point $c(g_0)$ by the map g_0) is compactly contained in V_0 . If, for a subsequence $n_k \rightarrow \infty$, $\lim_{k \rightarrow \infty} \frac{1}{n_k} \log |Dg_0^{n_k}(c(g_0))| = 0$, then also $\lim_{k \rightarrow \infty} \frac{1}{n_k} \log |Dg_1^{n_k}(c(g_1))| = 0$.*

Proof. Normalize the maps in such a way that $c(g_0) = c(g_1) = 0$ and $h(1) = 1$. As in [2], one can include g_0 and g_1 in a family g_ν of quasi-conformally conjugated maps of the class \mathcal{U} , with $c(g_\nu) = 0$, which depends holomorphically on $\nu \in D_r = \{|\nu| < r\}$, for some $r > 1$. Namely, if $\mu = \frac{\partial h}{\partial \bar{z}} / \frac{\partial h}{\partial z}$ is complex dilatation of h , then, for every $\nu \in D_r$, where $r = \|\mu\|_\infty^{-1}$, let h_ν be the unique q-c homeomorphism of \mathbb{C} with complex dilatation $\nu\mu$, which leaves the points 0, 1 fixed (in particular, $h_0 = id$ and $h_1 = h$). Then we can define domains $V_\nu = h_\nu(V_0)$, $V'_\nu = h_\nu(V'_0)$, and the map $g_\nu = h_\nu \circ g_0 \circ h_\nu^{-1} : V_\nu \rightarrow V'_\nu \in \mathcal{U}$ with $c(g_\nu) = 0$. As $\omega_{g_\nu}(0) = h_\nu(\omega_{g_0}(c(g_0)))$ is compactly contained in $V_\nu = h_\nu(V_0)$, by the Schwarz lemma and a compactness argument, given a compact subset K of the disk D_r , there exists C , such that $|Dg_\nu(g_\nu^i(0))| \leq C$ for every

$\nu \in K$ and every $i \geq 0$. Then $u_k(\nu) = n_k^{-1} \log |Dg_\nu^{n_k}(0)|$ is a sequence of harmonic functions in D_r , which is bounded on compacts. On the other hand, by Theorem 1.3, every limit value of the sequence $\{u_k\}$ is non-negative, and, by the assumption, $u_k(0) \rightarrow 0$. According to the Minimum Principle, $u_k(\nu) \rightarrow 0$ for any ν . \square

Remark 3.4. In particular, the Collet-Eckmann condition $\chi_-(g, c(g)) > 0$ is a quasi-conformal invariant. In fact, it is even a topological invariant [5].

Remark 3.5. Let $f(z) = z^d + c$, where $d \geq 2$ and $c \in \mathbb{C} \setminus \{0\}$, and let $\bar{x} = \{x_{-n}\}_{n=0}^\infty$, $x_0 = 0$, $f(x_{-n}) = x_{-(n-1)}$, $n > 0$, be a backward orbit of 0. Assume that $x_{-n} \neq 0$ for $n > 0$. Then

$$\chi_-^{back}(f, \bar{x}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x_{-n})| \geq 0.$$

The proof follows the one of Theorem 1.3 and uses a variant of Lemmas 2.1-2.2.

Some motivations and historical remarks.

In 1-dimensional dynamics often the asymptotic behaviour of derivatives along typical trajectories is reflected in the asymptotic behaviour of derivatives along critical trajectories. E.g. hyperbolicity is equivalent to the attraction of all critical trajectories to attracting periodic orbits. This implies $\chi_-(c) < 0$ for all critical values c . This paper provides converse theorems.

Another theory is that the ‘strong non-uniform hyperbolicity condition’ saying that there is $\chi > 0$ such that for all probability invariant measures μ on Julia set J for a rational map g $\chi_\mu(g) := \int \log |g'| d\mu \geq \chi$, is equivalent to so called Topological Collet-Eckmann condition see, see e.g. [6]. The latter in presence of only one critical point in J (and no parabolic orbits) is equivalent to Collet-Eckmann condition, see Remark 3.4.

A motivation to Theorem 1.1 has been the theorem saying that for all μ as above $\chi_\mu(g) \geq 0$, see [4]. In particular for μ -almost every $x \in J$, for any μ , $\chi(g, x) \geq 0$. This suggested the question whether critical values also have this property (under appropriate assumptions).

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